Dynamic Stability of Nash-Efficient Public Goods Mechanisms: Reconciling Theory and Experiments*

Yan Chen
Department of Economics, University of Michigan, Ann Arbor, MI 48109-1220
Phone:(734)763-9619; Fax: (734)764-2769; Email: yanchen@umich.edu

Revised: March 6, 2000

*I thank John Ledyard, David Roth and Tatsuyoshi Saijo for discussions that lead to this project; Klaus Abbink, Beth Allen, Rachel Croson, Roger Gordon, Elisabeth Hoffman, Matthew Jackson, Wolfgang Lorenzon, Laura Razzolini, Sara Solnick, Tayfun Sönmez, William Thomson, Lise Vesterlund, Xavier Vives, anonymous referees and an associate editor, seminar participants in Bonn (EDP), Hamburg, Michigan, Minnesota, Pittsburgh, Purdue, and participants of the 1997 North America Econometric Society Summer Meetings (Pasadena, CA), the 1997 Economic Science Association meetings (Tucson, AZ), the 1998 Midwest Economic Theory meetings (Ann Arbor, MI) and the 1999 NBER Decentralization Conference (New York, NY) for their comments and suggestions. The hospitality of the Wirtschaftspolitische Abteilung at the University of Bonn, the research support provided by Deutsche Forschungsgemeinschaft through SFB303 at the University of Bonn and NSF grant SBR-9805586 are gratefully acknowledged. Any remaining errors are my own.

Proposed running head: Dynamic Stability of Mechanisms

Mailing address of the author:
Yan Chen
Department of Economics
University of Michigan
611 Tappan Street
Ann Arbor, MI 48109-1220

Abstract

We propose to use supermodularity as a robust dynamic stability criterion for public goods mechanisms with a unique Nash equilibrium. Among the existing public goods mechanisms whose Nash equilibria are Pareto efficient, the Groves-Ledyard mechanism is a supermodular game if and only if the punishment parameter is sufficiently high, while none of the Hurwicz, Walker and Kim mechanisms is supermodular in a quasilinear environment. The Falkinger mechanism is a supermodular game in a quadratic environment if and only if the subsidy coefficient is greater than or equal to one. These results are consistent with the experimental findings in Smith (1979), Harstad and Marrese (1982), Mori (1989), Chen and Plott (1996), Chen and Tang (1998), and Falkinger, Fehr, Gächter and Winter-Ebmer (1998).

Keywords: public goods mechanisms, supermodular games, experiments
JEL Classification: H41, C62, D83

1 Introduction

The design of decentralized institutions to provide public goods has been a challenging problem for economists for a long time. Since the 1970s, economists have been seeking informationally decentralized mechanisms (i.e., mechanisms which only use the agents' own messages) that are non-manipulable (or dominant strategy incentive-compatible) and achieve a Pareto optimal allocation of resources with public goods.

By now it is well known that it is impossible to design a mechanism for making collective allocation decisions, which is informationally decentralized, non-manipulable, and Pareto optimal[1]. There are many mechanisms which preserve Pareto optimality at the cost of non-manipulability, some of which preserve "some degree" of non-manipulability. In particular, some mechanisms have been discovered which have the property that Nash equilibria are Pareto optimal[2]. These can be found in the work of Groves and Ledyard (1977), Hurwicz (1979), Walker (1981), Tian (1989), Kim (1993), Peleg (1996) and Falkinger (1996).

So far Nash implementation theory has mainly focused on establishing static properties of the equilibria. When a mechanism is implemented among real people, i.e., boundedly rational agents, however, we expect the actual implementation to be a dynamic process, starting somewhere off the equilibrium path[3]. Following Hurwicz (1972), one could interpret the Nash equilibrium strategies of a game form as the stationary messages of some decentralized learning process. The fundamental question concerning implementation of a specific mechanism is whether the dynamic processes will actually converge to one of the equilibria promised by theory. This paper addresses this question by proposing supermodularity as a robust stability criterion for public goods mechanisms when there is a unique Nash equilibrium.

The few theoretical papers on the dynamic properties of public goods mechanisms have been using very specific learning dynamics to investigate the stability of mechanisms. Muench and Walker (1983) and de Trenqualye (1988) study the convergence of the Groves-Ledyard

[1]This impossibility has been demonstrated in the work of Green and Laffont (1977), Hurwicz (1975), Roberts (1979) and Walker (1980) in the context of resource allocation with public goods.

[2]Other implementation concepts include perfect Nash equilibrium (Bagnoli and Lipman (1989)), undominated Nash equilibrium (Jackson and Moulin (1991)), etc.

[3]It is also possible to have instant realization of the equilibrium state, if there exists some agent who can compute the Nash equilibrium and recommend it to the other agents who then realize the wisdom of the recommendation and follow it. There are mixed experimental evidence for this recommendation method (Croson and Marks (1999)).

mechanism under Cournot best-reply dynamics. De Trenqualye (1989) and Vega-Redondo (1989) propose mechanisms for which the Cournot best-reply dynamics is globally convergent to the Lindahl equilibrium[4] outcome. Kim (1993) proposed a mechanism which implements Lindahl allocations and remains stable under the gradient adjustment process given quasi-linear utility functions. One exception is Cabrales (1999) who studies dynamic convergence and stability of the canonical mechanism in Nash implementation and the Abreu-Matsushima mechanism under "naive adaptive dynamics"[5].

Recent experimental studies on learning strongly reject the Cournot best-reply learning model in favor of other models (e.g., Boylan and El-Gamal (1993)). So far there has been no experimental investigation of the gradient adjustment process, even though it has been used fairly extensively in the theoretical research on stability of games (Arrow and Hurwicz (1977)). Experimental research on learning is still far from reaching a conclusion with regard to a single "true" learning model that describes all adaptive behaviors. Furthermore, there is strong evidence that individual players adopt different learning rules under different circumstances (El-Gamal and Grether (1995)). It is therefore desirable to identify mechanisms which converge under a wide class of learning dynamics. This paper does so by focusing on mechanisms which are supermodular games.

The class of supermodular games[6] has been identified as having very robust dynamic stability properties (Milgrom and Roberts (1990)): it converges to the set of Nash equilibria that bound the serially undominated set under a wide class of interesting learning dynamics, including Bayesian learning, fictitious play, adaptive learning, Cournot best-reply and many others. Therefore, instead of using a specific learning dynamic, we investigate whether we can find Nash-efficient public goods mechanisms which are supermodular games.

The idea of using supermodularity as a robust stability criterion for Nash-efficient mechanisms is not only based on its good theoretical properties, but also on strong experimental evidence. In fact it is inspired by the experimental results of Chen and Plott (1996) and Chen and Tang (1998), where they varied a punishment parameter in the Groves-Ledyard mechanism in a set of experiments and obtained totally different dynamic stability results.

In this paper, we review the main experimental findings on the dynamic stability of Nash-

[4] A Lindahl equilibrium for the public goods economy is characterized by a set of personalized prices and an allocation such that utility and profit maximization and feasibility conditions are satisfied.

[5] This is different from the adaptive learning in Milgrom and Roberts (1990). For a precise definition see Cabrales (1999).

[6] See Section 4 for a formal definition.

efficient public goods mechanisms, examine the supermodularity of existing Nash-efficient public goods mechanisms, and use the results to sort a class of experimental findings.

Section 2 introduces the environment. Section 3 reviews the experimental results. Section 4 discusses supermodular games. Section 5 investigates whether the existing mechanisms are supermodular games. Section 6 concludes the paper.

2 A public goods environment

We first introduce notation and the economic environment. Most of the experimental implementations of incentive-compatible mechanisms use a simple environment. Usually there is one private good x, one public good y, and $n \geq 3$ players, indexed by subscript i. Production technology for the public good exhibits constant returns to scale, i.e., the production function $f(\cdot)$ is given by $y = f(x) = x/b$ for some $b > 0$. Preferences are largely restricted to the class of quasilinear preferences[7]. Let E represent the set of transitive, complete and convex individual preference orderings, $\succeq_i$, and initial endowments, ω_i^x. We formally define E^Q as follows.

DEFINITION 1 $E^Q = \{(\succeq_i, \omega_i^x) \in E : \succeq_i$ *is representable by a* C^2 *utility function of the form* $v_i(y) + x_i$ *such that* $Dv_i(y) > 0$ *and* $D^2 v_i(y) < 0$ *for all* $y > 0$, *and* $\omega_i^x > 0\}$, *where* D^k *is the* k^{th} *order derivative.*

Falkinger, Fehr, Gächter and Winter-Ebmer (1998) use a quadratic environment in their experimental study of the Falkinger mechanism. We define this environment as E^{QD}.

DEFINITION 2 $E^{QD} = \{(\succeq_i, \omega_i^x) \in E : \succeq_i$ *is representable by a* C^2 *utility function of the form* $A_i x_i - \frac{1}{2} B_i x_i^2 + y$ *where* $A_i, B_i > 0$ *and* $\omega_i^x > 0\}$.

An *economic mechanism* is defined as a non-cooperative game form played by the agents. The game is described in its normal form. In all mechanisms considered in this paper, the implementation concept used is Nash equilibrium. In the Nash implementation framework the agents are assumed to have complete information about the environment while the designer does not know anything about the environment.

[7] Harstad and Marrese (1982) and Falkinger et. al. (1998) are exceptions.

3 Experimental results

Seven experiments have been conducted with mechanisms having Pareto-optimal Nash equilibria in public goods environments (see Chen (1999) for a survey). Sometimes the data converged quickly to the Nash equilibria; other times it did not. Smith (1979) studies a simplified version of the Groves-Ledyard mechanism which balanced the budget only in equilibrium. In the five-subject treatment (R1) one out of three sessions converged to the stage game Nash equilibrium. In the eight-subject treatment (R2) neither session converged to the Nash equilibrium prediction. Harstad and Marrese (1981) found that only three out of twelve sessions attained approximately Nash equilibrium outcomes under the simplified version of the Groves-Ledyard mechanism. Harstad and Marrese (1982) studied the complete version of the Groves-Ledyard mechanism in Cobb-Douglas economies. In the three-subject treatment one out of five sessions converged to the Nash equilibrium. In the four-subject treatment one out of four sessions converged to one of the Nash equilibria. Mori (1989) compares the performance of a Lindahl process with the Groves-Ledyard mechanism. He ran five sessions for each mechanism, with five subjects in each session. The aggregate levels of public goods provided in each of the Groves-Ledyard sessions were much closer to the Pareto optimal level than those provided using a Lindahl process. On the individual level, each of the five sessions stopped within ten rounds when every subject repeated the same messages. However, since individual messages must be in multiples of .25 while the equilibrium messages were not on the grid, convergence to Nash equilibrium messages was approximate. None of the above experiments studied the effects of the punishment parameter[8] on the performance of the mechanism.

Chen and Plott (1996) first assessed the performance of the Groves-Ledyard mechanism under different punishment parameters. Each group consisted of five players with different preferences. They found that by varying the punishment parameter the dynamics and stability changed dramatically. This finding was replicated by Chen and Tang (1998) with twenty-one independent sessions and a longer time series (100 rounds) in an experiment designed to study the learning dynamics. Chen and Tang (1998) also studied the Walker mechanism[9] in the same economic environment.

[8]See Section 5 for a formal definition. Roughly speaking, the punishment parameter in the Groves-Ledyard mechanism determines the magnitude of punishment if a player's contribution deviates from the mean of other players' contributions.

[9]See Walker (1981). A formal definition of the Walker mechanism is provided in the Appendix.

[Figure 1 about here.]

Figures 1 presents the time series data from Chen and Tang (1998) for two out of five types of players[10]. Each graph presents the mean (the black dots), standard deviation (the error bars) and stage game equilibria (the dashed lines) for each of the two different types averaged over seven independent sessions for each mechanism. The two graphs in the first column display the mean contribution (and standard deviation) for type 1 and 2 players under the Walker mechanism (hereafter Walker). The second column displays the average contributions for type 1 and 2 for the Groves-Ledyard mechanism under a low punishment parameter (hereafter GL1). The third column displays the same information for the Groves-Ledyard mechanism under a high punishment parameter (hereafter GL100). From these graphs, it is apparent that all seven sessions of the Groves-Ledyard mechanism under a high punishment parameter converged[11] very quickly to its stage game Nash equilibrium and remained stable, while the same mechanism did not converge under a low punishment parameter; the Walker mechanism (Walker) did not converge to its stage game Nash equilibrium either.

Because of its good dynamic properties, GL100 had far better performance than GL1 and Walker, evaluated in terms of system efficiency, close to Pareto optimal level of public goods provision, less violations of individual rationality constraints and convergence to its stage game equilibrium. All these results are statistically highly significant (Chen and Tang (1998)).

These results illustrate the importance to design mechanisms which not only have good static properties, but also good dynamic stability properties like GL100. Only when the dynamics lead to the convergence to the static equilibrium, can all the nice static properties be realized.

Falkinger, Fehr, Gächter and Winter-Ebmer (1998) studies the Falkinger mechanism in a quasilinear as well as a quadratic environment. In the quasilinear environment, the

[10] The data for the remaining three types of players are not displayed due to limited space, but are available from the author upon requests. They display very similar patterns.

[11] "Theoretically, convergence implies that no deviation will ever be observed once the system equilibrates. In an experimental setting with long iterations, even after the system equilibrates, subjects sometimes experiment by occasional deviation. Therefore, it is necessary to have some behavioral definition of convergence: a system converges to an equilibrium at round , if $_i( ) = {}^e_i$, $\forall$ and $\forall \; \geq$, except for a maximum of rounds of deviation for , where is small. For our experiments of 100 rounds, we let ≤ 5, i.e., there could be a total of up to 5 rounds of experimentation or mistakes after the system converged." Chen and Tang (1998).

mean contributions moved towards the Nash equilibrium level but did not quite reach the equilibrium. In the quadratic environment the mean contribution level hovered around the Nash equilibrium, even though none of the 23 sessions had a mean contribution level exactly equal to the Nash equilibrium level in the last five rounds. Therefore, Nash equilibrium was a good description of the average contribution pattern, although individual players did not necessarily play the equilibrium.

In Section 5 we will provide a theoretical explanation for the above experimental results in light of supermodular games.

4 Supermodularity and stability

We first define supermodular games and review their stability properties. Then we discuss alternative stability criteria and their relationship with supermodularity.

Supermodular games are games in which each player's marginal utility of increasing her strategy rises with increases in her rival's strategies, so that (roughly) the player's strategies are "strategic complements". Supermodular games need an order structure on strategy spaces, a weak continuity requirement on payoffs, and complementarity between components of a player's own strategies, in addition to the above-mentioned strategic complementarity between players' strategies. Suppose each player i's strategy set S_i is a subset of a finite-dimensional Euclidean space $\mathbb{R}^{k_i}$. Then $S \equiv \times_{i=1}^{n} S_i$ is a subset of $\mathbb{R}^k$, where $k = \sum_{i=1}^{n} k_i$.

DEFINITION 3 *A* supermodular game *is such that, for each player i, S_i is a nonempty sublattice of $\mathbb{R}^{k_i}$, u_i is upper semi-continuous in s_i for fixed s_{-i} and continuous in s_{-i} for fixed s_i, u_i has increasing differences in (s_i, s_{-i}), and u_i is supermodular in s_i.*

Increasing differences says that an increase in the strategy of player i's rivals raises her marginal utility of playing a high strategy. The supermodularity assumption ensures complementarity among components of a player's own strategies. Note it is automatically satisfied when S_i is one-dimensional. As the following theorem indicates supermodularity and increasing differences are easily characterized for smooth functions in $\mathbb{R}^n$.

THEOREM 1 (Topkis (1978)) *Let u_i be twice continuously differentiable on S_i. Then u_i has increasing differences in (s_i, s_j) if and only if $\partial^2 u_i / \partial s_{ih} \partial s_{jl} \geq 0$ for all $i \neq j$ and all $1 \leq h \leq k_i$ and all $1 \leq l \leq k_j$; and u_i is supermodular in s_i if and only if $\partial^2 u_i / \partial s_{ih} \partial s_{il} \geq 0$ for all i and all $1 \leq h < l \leq k_i$;*

Supermodular games are of interest particularly because of their very robust stability properties. Milgrom and Roberts (1990) proved that in these games the set of learning algorithms consistent with adaptive learning converge to the set bounded by the largest and the smallest Nash equilibrium strategy profiles. Intuitively a sequence is consistent with adaptive learning if players "eventually abandon strategies that perform consistently badly in the sense that there exists some other strategy that performs strictly and uniformly better against every combination of what the competitors have played in the not too distant past"[12]. This includes a wide class of interesting learning dynamics, such as Bayesian learning, fictitious play, adaptive learning, Cournot best-reply and many others.

Since experimental evidence suggests that individual players tend to adopt different learning rules (El-Gamal and Grether (1995)), instead of using a specific learning algorithm to study stability, one can use supermodularity as a robust stability criterion for games with a unique Nash equilibrium. For supermodular games with a unique Nash equilibrium, we expect any adaptive learning algorithm to converge to the unique Nash equilibrium, in particular, Cournot best-reply, fictitious play and adaptive learning. Compared with stability analysis using Cournot best-reply dynamics, supermodularity is much more robust and inclusive in the sense that it implies stability under Cournot best-reply and many other learning dynamics mentioned above. Among the seven experiments in Nash-efficient public goods mechanisms which we discussed in Section 3, six of them has a unique Nash equilibrium. The only one with multiple Nash equilibria is Harstad and Marrese (1982).

There are two caveats for using supermodularity as a robust stability criterion. First, for supermodular games with multiple Nash equilibria, adaptive learning algorithms will converge to the set bounded by the largest and the smallest Nash equilibrium strategy profiles, however, players might not be able to learn to coordinate on a particular equilibrium. Van Huyck, Battalio and Beil (1990) examine a finitely repeated coordination game with seven Nash equilibria, which is supermodular. They found that with 14 to 16 players play tended to converge to the Pareto-dominated Nash equilibrium where each player chooses the minimum effort level. With two players, however, for 12 out of 14 pairs play converged to the Pareto-dominant Nash equilibrium where each player chooses the maximum effort level. Therefore, the equilibrium selection problem might depend on the group size and many other factors. Supermodularity does not help to predict which equilibrium will be selected.

Second, supermodularity is sufficient but not necessary for convergence. This implies

[12] For a formal definition, see Milgrom and Roberts (1990).

that supermodular mechanisms with a unique Nash equilibrium ought to converge to the Nash equilibrium prediction fairly robustly, but mechanisms which are not supermodular could still converge to its equilibrium under some learning algorithms. In particular, supermodular games with a unique pure strategy Nash equilibrium is dominance solvable, but not vice versa. The robust convergence argument for supermodular games also applies to the larger class of dominance solvable games (Milgrom and Roberts (1991)). Dominance solvability is more inclusive but harder to check than supermodularity. There have been some experiments, which support dominance solvability as the outcome of adaptive learning. Chen (1998) studies the serial and average cost pricing mechanism in an environment where both mechanisms are dominance solvable. Under complete information, more than 80 percent of the players converged to the unique Nash equilibrium within eight rounds under both mechanisms. Cox and Walker (1998) study whether subjects can learn to play Cournot duopoly strategies in games with two kinds of interior Nash equilibrium. Their type I duopoly has a stable interior Nash equilibrium under Cournot best-reply dynamics and therefore is dominance solvable[13]. Their type II duopoly has an unstable interior Nash equilibrium and two boundary equilibria under Cournot best-reply dynamics, and therefore is not dominance solvable. They found that after a few periods subjects did play stable interior, dominance solvable equilibria, but they did not play the unstable interior equilibria nor the boundary equilibria.

For a complete characterization of the dynamic stability of mechanisms, it is desirable to find both sufficient and necessary conditions for convergence under a wide range of learning dynamics. Since learning can differ from one context to another, we would need extensive experimental studies of human learning behavior under different mechanisms and the resulting repertoire of algorithms, calibrated against human responses, to cover various contexts. This is largely still an ongoing research enterprise. Once we have the accurately calibrated algorithms, we can restrict ourselves to the stability analysis based on these algorithms, and perhaps eventually characterize the sufficient and necessary conditions for these learning dynamics to converge.

[13] Moulin (1984) shows that dominance solvability implies Cournot stability, but the converse need not hold in general. The converse does hold for two-player games with one-dimensional strategy space and strictly quasiconcave payoffs, which applies to the Cox and Walker experiments.

5 Supermodularity of existing Nash-efficient public goods mechanisms

In this section we investigate the supermodularity of five well-known Nash-efficient public goods mechanisms. We use supermodularity to analyze the experimental results on Nash-efficient public goods mechanisms.

The Groves-Ledyard mechanism (1977) is the first mechanism in a general equilibrium setting whose Nash equilibrium is Pareto optimal. The mechanism allocates private goods through the competitive markets and public goods through a government allocation-taxation scheme that depends on information communicated to the government by consumers regarding their preferences. Given the government scheme, consumers find it in their best interest to reveal their true preferences for public goods. The mechanism balances the budget both on and off the equilibrium path, but it does not implement Lindahl allocations. Later on, more game forms have been discovered which implement Lindahl allocations in Nash equilibrium. These include Hurwicz (1979), Walker (1981), Tian (1989), Kim (1993) and Peleg (1996).

DEFINITION 4 *For the Groves-Ledyard mechanism, the strategy space of player i is $S_i \subset I\!R^1$ with generic element $m_i \in S_i$. The outcome function of the public good and the net cost share of the private good for player i are*

$$\begin{aligned} Y(m) &= \sum_k m_k \\ T_i^{GL}(m) &= \frac{Y(m)}{n} \cdot b + \frac{\gamma}{2}\left[\frac{n-1}{n}(m_i - \mu_{-i})^2 - \sigma_{-i}^2\right]. \end{aligned}$$

where $\gamma > 0$, $n \geq 3$, $\mu_{-i} = \sum_{j \neq i} m_j/(n-1)$ is the mean of others' messages, and $\sigma_{-i}^2 = \sum_{h \neq i}(m_h - \mu_{-i})^2/(n-2)$ is the squared standard error of the mean of others' messages.

In the Groves-Ledyard mechanism each agent reports m_i, the increment (or decrement) of the public good player i would like to add to (or subtract from) the amounts proposed by others. The planner sums up the individual contributions to get the total amount of public good, Y, and taxes each individual based on her own message, and the mean and sample variance of everyone else's messages. Thus each individual's tax share is composed of three parts: the per capita cost of production, $Y \cdot b/n$, plus a positive multiple, $\gamma/2$, of the difference between her own message and the mean of others' messages, $((n-1)/n) \times (m_i - \mu_{-i})^2$, and the sample variance of others' messages, σ^2_{-i}. While the first two parts guarantee that Nash equilibria of the mechanism are Pareto optimal, the last part insures that budget is balanced both on and off the equilibrium path. Note that the free parameter, γ, determines the magnitude of punishment when an individual deviates from the mean of others' messages. It does not affect any of the static theoretical properties of the mechanism.

Chen and Plott (1996) and Chen and Tang (1998) found that the punishment parameter, γ, had a significant effect in inducing convergence and dynamic stability. For a large enough γ, the system converged to its stage game Nash equilibrium very quickly and remained stable; while under a small γ, the system did not converge to its stage game Nash equilibrium. In the following proposition, we provide a necessary and sufficient condition for the mechanism to be a supermodular game given quasilinear preferences, and thus to converge to its Nash equilibrium under a wide class of learning dynamics.

PROPOSITION 1 *The Groves-Ledyard mechanism is a supermodular game for any* $e \in E^Q$ *if and only if* $\gamma \in [-\min_{i \in N}\{\frac{\partial\ v_i}{\partial y}\}n, +\infty)$.

Proof: **(i)** If $\gamma \in [-\min_{i \in N}\{\frac{\partial\ v_i}{\partial y}\}n, +\infty)$, then the Groves-Ledyard mechanism is a supermodular game for any $e \in E^Q$: see Chen and Tang (1998).

(ii) Next, we prove that if the Groves-Ledyard mechanism is a supermodular game for any $e \in E^Q$, then $\gamma \in [-\min_{i \in N}\{\frac{\partial\ v_i}{\partial y}\}n, +\infty)$. Since it is a supermodular game, the payoff function, u_i, has increasing differences in (m_i, m_{-i}), for all i. Since u_i is C^2 on S_i, by Theorem 1, u_i has increasing differences in (m_i, m_{-i}) if and only if

$$\frac{\partial^2 u_i}{\partial m_i \partial m_j} = \frac{\partial^2 v_i}{\partial y^2} + \gamma/n \geq 0, \forall i,$$

which implies that $\gamma \in [-\min_{i \in N}\{\frac{\partial\ v_i}{\partial y}\}n, +\infty)$. *Q.E.D.*

Therefore, when the punishment parameter is above the threshold, a large class of interesting learning dynamics converge, which is consistent with the experimental results. Intuitively, when the punishment parameter is sufficiently high, the incentive for each agent to match the mean of other agents' messages is also high. Therefore, when other agents increase their contributions, agent i also wants to increase her contribution to avoid the penalty. Thus the messages become strategic complements and the game is transformed into a supermodular game. Muench and Walker (1983) found a convergence condition for the

Groves-Ledyard mechanism using Cournot best-reply dynamics and parameterized quadratic preferences. This proposition generalizes their result to general quasilinear preferences and a much wider class of learning dynamics.

Falkinger (1996) introduces a class of simple mechanisms. In this incentive compatible mechanism for public goods, Nash equilibrium is Pareto optimal when a parameter is chosen appropriately[14], however, it does not implement Lindahl allocations and the existence of equilibrium can be delicate in some environments.

DEFINITION 5 *For the Falkinger (1996) mechanism, the strategy space of player i is $S_i \subset I\!R^1$ with generic element $m_i \in S_i$. The outcome function of the public good and the net cost share of the private good for player i are*

$$\begin{aligned} Y(m) &= \sum_k m_k, \\ T_i^F(m) &= b[m_i - \beta(m_i - \frac{\sum_{j \neq i} m_j}{n-1})], \end{aligned}$$

where $\beta > 0$.

This tax-subsidy scheme works as follows: if an individual's contribution is above the average contribution of the others, she gets a subsidy of β for a marginal increase in her contribution. If her contribution is below the average contribution of others, she has to pay a tax whereby a marginal increase in her contribution reduces her tax payment by β. If β is chosen appropriately, Nash equilibrium of this mechanism is Pareto efficient. Furthermore, it fully balances the budget in and out of equilibrium path.

PROPOSITION 2 *The Falkinger mechanism is a supermodular game for any $e \in E^{QD}$ if and only if $\beta \geq 1$.*

Proof: **(i)** If $\beta \geq 1$, then the Falkinger mechanism is a supermodular game for any $e \in E^{QD}$.

For any $e \in E^{QD}$, the continuity condition is satisfied. Since $S_i \subset \mathbb{R}^1$, the supermodularity condition is automatically satisfied. If $\beta \geq 1$, then

$$\frac{\partial^2 u_i}{\partial m_i \partial m_j} = \frac{B_i b^2}{n-1}\beta(\beta - 1) \geq 0, \forall i.$$

[14] In this mechanism, when $\quad = 1 - 1 \quad$, Nash equilibrium is Pareto optimal.

By Theorem 1, u_i has increasing differences in (m_i, m_{-i}), for all i. Therefore, it is a supermodular game.

(ii) Next, we prove that if the Falkinger mechanism is a supermodular game for any $e \in E^{QD}$, then $\beta \geq 1$.

Since it is a supermodular game, the payoff function, u_i, has increasing differences in (m_i, m_{-i}), for all i. Since u_i is C^2 on S_i, by Theorem 1, u_i has increasing differences in (m_i, m_{-i}) if and only if

$$\frac{\partial^2 u_i}{\partial m_i \partial m_j} = \frac{B_i b^2}{n-1} \beta(\beta - 1) \geq 0, \forall i.$$

which implies that $\beta \geq 1$. *Q.E.D.*

Since Pareto efficiency requires that $\beta = 1 - 1/n$, in a large economy, this will produce a game which is close to being a supermodular game. It is interesting to note that in the quadratic environment of Falkinger, Fehr, Gächter and Winter-Ebmer (1998), the game is very close to being a supermodular game: in the experiment β was set to 2/3. The results show the mean contribution level hovered around the Nash equilibrium, even though none of the 23 sessions had a mean contribution level exactly equal to the Nash equilibrium level in the last five rounds. Their results suggest that the convergence in supermodular games might be a function of the degree of strategic complementarity. That is, in games with a unique Nash equilibrium which can induce supermodular games, as the degree of strategic complementarity increases, we might observe increased convergence to its stage game Nash equilibrium.

Three specific game forms[15] implementing Lindahl allocations in Nash equilibrium have been introduced, Hurwicz (1979), Walker (1981), and Kim (1993). All three improve on the Groves-Ledyard mechanism in the sense that they all satisfy the individual rationality constraint in equilibrium. While Hurwicz (1979) and Walker (1981) can be shown to be unstable for any decentralized adjustment process in certain quadratic environments (Kim (1986)), the Kim mechanism is stable under a gradient adjustment process given quasilinear utility functions, which is a continuous time version of the Cournot-Nash tatonnement adjustment process. Whether the Kim mechanism is stable under other decentralized learning processes is still an open question. Kim (1986) has shown that for any game form implementing Lin-

[15] Since Tian (1989) and Peleg (1996) do not have specific mechanisms, we will only investigate the supermodularity of these three mechanisms.

dahl allocations there does not exist a decentralized adjustment process which ensures local stability of Nash equilibria in certain classes of environments.

PROPOSITION 3 *None of the Hurwicz (1979), Walker (1981) and Kim (1993) mechanisms is a supermodular game for any $e \in E^Q$.*

Proof: See Appendix. ∎

The following observation organizes all experimental results on Nash-efficient public goods mechanisms with available parameters[16] by looking at whether they are supermodular games.

OBSERVATION 1 *(1) None of the following experiments is a supermodular game: the Groves-Ledyard mechanism studied in Smith's (1979) R2 treatment, Harstad and Marrese (1982), Mori (1989), Chen and Plott (1996)'s low γ treatment, and Chen and Tang (1998)'s low γ treatment, the Walker mechanism in Chen and Tang (1998), and the Falkinger mechanism in Falkinger, Fehr, Gächter and Winter-Ebmer (1998).*
(2) The Groves-Ledyard mechanism under the high γ in Chen and Plott (1996) and Chen and Tang (1998) are both supermodular games.

Therefore, none of the existing experiments which did not converge is a supermodular game, while those which did converge well are both supermodular games.

Note that none of the above experiments was designed to study supermodular games. They were designed to study other issues but serendipitously studied supermodular games. These studies revealed some very interesting results as well as important open questions where theory is silent. In particular, the theory on the stability of supermodular games is silent with regard to the degree of strategic complementarity on convergence. The results from Falkinger, Fehr, Gächter and Winter-Ebmer (1998) seems to suggest the following hypothesis:

HYPOTHESIS 1 *In games with a unique Nash equilibrium which can induce supermodular games, as the degree of strategic complementarity increases, we will observe increased convergence to its stage game Nash equilibrium.*

[16]The design parameters used in Smith's (1979) R1 treatment and Harstad and Marrese (1981) are not available.

Both the Groves-Ledyard and the Falkinger mechanism are good playground for studying Hypothesis 1. One can design an experiment which set the free parameter in each mechanism from below the threshold (to induce a supermodular game), then gradually increase it to exactly equal the threshold, and then gradually increase it above the threshold, with finer steps around the threshold. This will present a gradual increase in the degree of strategic complementarities with finer partitions around the threshold. This will allow the researcher to observe whether a gradual increase in the strategic complementarity will lead to a gradual increase in the degree of convergence.

6 Concluding Remarks

So far Nash implementation theory has mainly focused on establishing static properties of the equilibria. However, experimental evidence suggests that the fundamental question concerning any actual implementation of a specific mechanism is whether decentralized dynamic learning processes will actually converge to one of the equilibria promised by theory. Based on its attractive theoretical properties[17] and the supporting evidence for these properties in the experimental literature, we focus on supermodularity as a robust stability criterion for Nash-efficient public goods mechanisms with a unique Nash equilibrium.

This paper demonstrates that given a quasilinear utility function the Groves-Ledyard mechanism is a supermodular game if and only if the punishment parameter is above a certain threshold while none of the Hurwicz, Walker and Kim mechanisms is a supermodular game. The Falkinger mechanism can be converted into a supermodular game in a quadratic environment if the subsidy coefficient is above one. These results generalize a previous convergence result on the Groves-Ledyard mechanism by Muench and Walker (1983). They are consistent with the experimental findings of in Smith (1979), Harstad and Marrese (1982), Mori (1989), Chen and Plott (1996), Chen and Tang (1998), and Falkinger, Fehr, Gächter and Winter-Ebmer (1998).

Two aspects of the convergence and stability analysis in this paper warrant attention. First, supermodularity is sufficient but not necessary for convergence to hold. It is possible that a mechanism could fail supermodularity but still behaves well on a class of adjustment

[17] In particular, Milgrom and Roberts (1990) have shown that a supermodular game converges under a wide class of learning dynamics, including Bayesian learning, fictitious play, adaptive learning, Cournot best response and many others.

dynamics, such as the Kim mechanism. Secondly, The stability analysis in this paper, like other theoretical studies of the dynamic stability of Nash mechanisms, have been mostly restricted to quasilinear utility functions. It is desirable to extend the analysis to other more general environments. The maximal domain of stable environments remains an open question.

APPENDIX

Before proving Proposition 3, we first define the three mechanisms. All three mechanisms require that the number of players is at least three, i.e., $n \geq 3$.

DEFINITION 6 *For the Hurwicz (1979) mechanism, the strategy space of player i is $S_i \subset I\!R^2$ with generic element $(p_i, y_i) \in S_i$. The outcome function of the public good and the net cost share of the private good for player i are*

$$\begin{aligned} Y(y) &= \frac{\sum_k y_k}{n}, \\ T_i^H(p, y) &= R_i \cdot Y(y) + p_i \cdot (y_i - y_{i+1})^2 - p_{i+1} \cdot (y_{i+1} - y_{i+2})^2, \end{aligned}$$

where $R_i = \frac{1}{n} + p_{i+1} - p_{i+2}$.

DEFINITION 7 *For the Walker (1981) mechanism, the strategy space of player i is $S_i \subset I\!R^1$ with generic element $m_i \in S_i$. The outcome function of the public good and the net cost share of the private good for player i are*

$$\begin{aligned} Y(m) &= \sum_k m_k, \\ T_i^W(m) &= (\frac{1}{n} + m_{i-1} - m_{i+1}) \cdot Y(m). \end{aligned}$$

DEFINITION 8 *For the Kim (1993) mechanism, the strategy space of player i is $S_i \subset I\!R^2$ with generic element $(m_i, z_i) \in S_i$. The outcome function of the public good and the net cost share of the private good for player i are*

$$\begin{aligned} Y(m) &= \sum_k m_k, \\ T_i^K(m, z) &= P_i(m, z) \cdot Y(m) + \frac{1}{2}(z_i - \sum_k m_k)^2, \end{aligned}$$

where $P_i(m, z) = \frac{b}{n} - \sum_{j \neq i} m_j + \frac{1}{n} \sum_{j \neq i} z_j$.

Proof of Proposition 3:
(1) To show that the Hurwicz mechanism is not a supermodular game for any $e \in E^Q$, it suffices to show that the payoff function, u_i, does not have increasing difference in (s_i, s_{-i}).

Since $u_i(p, y) = v_i(y) + \omega_i - T_i^H$, we have

$$\frac{\partial^2 u_i}{\partial y_i \partial y_j} = \frac{1}{n^2} \frac{\partial^2 v_i}{\partial y^2}, \text{ for all } j \neq i+1.$$

By Definition 1, $\frac{\partial \ v_i}{\partial y} < 0$, so

$$\frac{\partial^2 u_i}{\partial y_i \partial y_j} < 0, \text{ for all } i \text{ and for all} j \neq i+1.$$

By Theorem 1, u_i does not have increasing difference in (s_i, s_{-i}).

(2) To show that the Walker mechanism is not a supermodular game for any $e \in E^Q$, it suffices to show that the payoff function, u_i, does not have increasing difference in (m_i, m_{-i}):

$$\frac{\partial^2 u_i}{\partial m_i \partial m_j} = \begin{cases} \frac{\partial \ v_i}{\partial y} + 1, & \text{if } j = i+1; \\ \frac{\partial \ v_i}{\partial y} - 1, & \text{if } j = i-1; \\ \frac{\partial \ v_i}{\partial y}, & \text{if } j \neq i-1, i+1. \end{cases}$$

By Definition 1, $\frac{\partial \ v_i}{\partial y} < 0$, so $\frac{\partial \ u_i}{\partial m_i \partial m_j} < 0$ for all $j \neq i+1$. By Theorem 1, u_i does not have increasing difference in (s_i, s_{-i}).

(3) To show that the Kim mechanism is not a supermodular game for any $e \in E^Q$, it suffices to show that the payoff function, u_i, does not have increasing difference in (s_i, s_{-i}):

$$\frac{\partial^2 u_i}{\partial m_i \partial z_j} = -\frac{1}{n} < 0.$$

By Theorem 1, u_i does not have increasing difference in (s_i, s_{-i}). Q.E.D.

References

[1] Abreu, D. and H. Matsushima (1992), "Virtual Implementation in Iteratively Undominated Strategies: Complete Information", *Econometrica* 60-3: 993-1008.

[2] Arrow, K. and L. Hurwicz (1977), "Stability of the Gradient Process in N Person Games", in K. Arrow and L. Hurwicz eds., *Studies in Resource Allocation Processes*: Cambridge University Press, Cambridge.

[3] Bagnoli, M. and B. Lipman (1989), "Provision of Public Goods: Fully Implementing the Core Through Private Contributions", *Review of Economic Studies* 56: 583-602.

[4] Boylan, R. and M. El-Gamal (1993), "Fictitious Play: A Statistical Study of Multiple Economic Experiments", *Games and Economic Behavior* 5, 205-222.

[5] Cabrales, A. (1999), "Adaptive Dynamics and the Implementation Problem with Complete Information", *Journal of Economic Theory* 86: 159-184.

[6] Chen, Y. (1998), "An Experimental Study of the Serial and Average Cost Pricing Mechanisms". Manuscript: University of Michigan.

[7] Chen, Y. (1999), "Incentive-Compatible Mechanisms for Pure Public Goods: A Survey of Experimental Research", in C. Plott and V. Smith (eds) *The Handbook of Experimental Economics Results*. Elsevier Press: Amsterdam, forthcoming.

[8] Chen, Y. and C. R. Plott (1996), "The Groves-Ledyard Mechanism: An Experimental Study of Institutional Design", *Journal of Public Economics* 59: 335-364.

[9] Chen, Y. and F.-F. Tang (1998), "Learning and Incentive-Compatible Mechanisms for Public Goods Provision: An Experimental Study", *Journal of Political Economy* 106: 633-662.

[10] Croson, R. and M. Marks (1999), "The Effect of Recommended Contributions in the Voluntary Provision of Public Goods", mimeo, University of Pennsylvania.

[11] Cox, J.C., M. Walker and J. Shachat (1996), "An Experimental Test of Bayesian vs. Adaptive Learning in Normal Form Games", mimeo, University of Arizona.

[12] Cox, J.C. and M. Walker (1998), "Learning to Play Cournot Duopoly Strategies", *Journal of Economic Behavior and Organization* 36: 141-161.

[13] El-Gamal, M. and D. Grether (1995), "Uncovering Behavioral Strategies: Are People Bayesians?", *Journal of the American Statistical Associations* 90: 1137-1145.

[14] Falkinger, J. (1996), "Efficient Private Provision of Public Goods by Rewarding Deviations from Average", *Journal of Public Economics* 62: 413-422.

[15] Falkinger, J., E. Fehr, S. Gächter and R. Winter-Ebmer (1998), "A Simple Mechanism for the Efficient Provision of Public Goods - Experimental Evidence". Forthcoming in *American Economic Review.*

[16] Green, J. and J.-J. Laffont (1977), "Characterization of Satisfactory Mechanisms for the Revelation of the Preferences for Public Goods", *Econometrica* 45: 427-438.

[17] Groves, T. and J. Ledyard (1977), "Optimal Allocation of Public Goods: A Solution to the 'Free Rider' Problem", *Econometrica* 45, no. 4, 783-809.

[18] Harstad, R. M., and M. Marrese (1981), "Implementation of Mechanism by Processes: Public Good Allocation Experiments", *Journal of Economic Behavior and Organization* 2: p.129 - 151.

[19] Harstad, R. M., and M. Marrese (1982), "Behavioral Explanations of Efficient Public Good Allocations", *Journal of Public Economics* 19: 367 - 383.

[20] Hurwicz, L. (1972), "On Informationally Decentralized Systems", *Decision and Organization* ed. C. McGuire and R. Radner (North Holland, Amsterdam): 297-336.

[21] Hurwicz, L. (1979), "Outcome Functions Yielding Walrasian and Lindahl Allocations at Nash Equilibrium Points", *Review of Economic Studies*: 217-225.

[22] Jackson, M. and H. Moulin (1992), "Implementing a Public Project and Distributing its Costs", *Journal of Economic Theory* 57, 125-140.

[23] Kim, T. (1986), "On the Nonexistence of a Stable Nash Mechanism implementing Lindahl Allocations", University of Minnesota mimeo.

[24] Kim, T. (1993), "A Stable Nash Mechanism Implementing Lindahl Allocations for Quasi-linear Environments", *Journal of Mathematical Economics* 22: 359-371.

[25] Ledyard, J. (1993), "Public Goods: A Survey of Experimental Research", in Kagel and Roth ed. *Handbook of Experimental Economics*: Princeton University Press.

[26] Milgrom, P., and J. Roberts (1990), "Rationalizability, Learning and Equilibrium in Games with Strategic Complementarities", *Econometrica* 58, No. 6: 1255-1277.

[27] Milgrom, P., and J. Roberts (1991), "Adaptive and Sophisticated Learning in Normal Form Games", *Games and Economic Behavior* 3: 82-100.

[28] Mori, T. (1989), "Effectiveness of Mechanisms for Public Goods Provision: An Experimental Study", *Economic Studies* 40 (3): 234-246.

[29] Muench, T., and M. Walker (1983), "Are Groves-Ledyard Equilibria Attainable?", *Review of Economic Studies* L, 393-396.

[30] Moulin, H. (1984), "Dominance Solvability and Cournot Stability", *Mathematical Social Sciences* 7: 83-102.

[31] Peleg, B. (1996), "Double Implementation of the Lindahl Equilibrium by a Continuous Mechanism", *Economic Design* 2: 311-324.

[32] Roberts, J. (1979), "Incentives and Planning Procedures for the Provision of Public Goods", *Review of Economic Studies* 46: 283-292.

[33] Smith, V. (1979), "Incentive Compatible Experimental Processes For the Provision of Public Goods", [ed.] by Smith *Research in Experimental Economics* vol 1, JAI Press Inc.: Greenwich, Connecticut.

[34] Tian, G. (1989), "Implementation of the Lindahl Correspondence by a Single-Valued, Feasible, and Continuous Mechanism", *Review of Economic Studies* 56: 613-621.

[35] Topkis, D. (1978), "Minimizing a Submodular Function on a Lattice", *Operations Research* 26: 305-321.

[36] de Trenqualye, P. (1988), "Stability of the Groves-Ledyard Mechanism", *Journal of Economic Theory* 46: 164-171.

[37] de Trenqualye, P. (1989), "Stable Implementation of Lindahl Allocations", *Economic Letters* 29: 291-294.

[38] Vega-Redondo, F. (1989), "Implementation of Lindahl Equilibrium: An Integration of Static and Dynamic Approaches", *Mathematical Social Sciences* 18: 211-228.

[39] Walker, M. (1980), "On the Impossibility of a Dominant Strategy Mechanism to Optimally Decide Public Questions", *Econometrica* 48: 1521- 1540.

[40] Walker, M. (1981), "A Simple Incentive Compatible Scheme for Attaining Lindahl Allocations", *Econometrica* 49: 65-71.